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LETTER TO THE EDITOR

The heat kernel for deformed spheres

N Shtykov†§ and D V Vassilevich‡

† Department of Physics, Tokyo Institute of Technology, Oh-Okayama, Meguro-ku, Tokyo, 152, Japan

‡ Department of Theoretical Physics, St Petersburg University, 198904 St Petersburg, Russia

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Abstract. We derive the asymptotic expansion of the heat kernel for a Laplace operator acting on deformed spheres. We calculate the coefficients of the heat kernel expansion on two- and three-dimensional deformed spheres as functions of deformation parameters. We find that under some deformation the conformal anomaly for free scalar fields on $R^4 \times \tilde{S}^2$ and $R^6 \times \tilde{S}^2$ is cancelled.

The asymptotic expansion of the heat kernel, corresponding to the elliptic second-order differential operator acting on an arbitrary manifold M has been investigated in connection with index theorems [1] and some applications in field theory [2, 3]. The kernel $K(x, y, t)$ satisfies a heat equation for some second-order operator $H = -D^2 + X$ defined on a smooth N-dimensional Riemannian manifold (X is a scalar function)

$$(\partial_t + H)K(x, y, t) = 0$$

with the boundary condition

$$K(x, y, 0) = \delta(x, y).$$

The asymptotic expansion of $K(x, y, t)$ has been derived for various models [4–7] in a general form [6] and in a numerical form for some homogeneous spaces [7]. Under $t \to 0$, the heat kernel has the following expansion:

$$K(x, y, t) = (4\pi t)^{-N/2}\Delta^{1/2}(x, y)\exp(-\sigma^2/4t)\sum_{n=0}^{\infty} a_n(x, y)(t)^n$$

where Δ is the invariant Van Vleck–Morette determinant [8] and $2\sigma(x, y)$ is the square of the geodesic distance between x and y. In terms of $K(x, y, t)$, one can write a simple integral representation for the one-loop effective action. If one takes regularization with the short-distance cut-off L [9], the regularized one-loop effective action $W^{(1)}$ can be defined as

$$W^{(1)} = \int_L^\infty \frac{dt}{t} K(t).$$

§ On leave from Irkutsk University, Russia; E-mail: shtykov@phys.titech.ac.jp

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Here $K(t) = \mathrm{tr}\int \mathrm{d}^N x g^{1/2} K(x,x,t)$, with the asymptotic expansion

$$K(t) = \sum_{n=0}^{\infty} A_n t^{n-N/2} = \sum_{n=0}^{\infty} \mathrm{tr} \int \mathrm{d}^N x g^{1/2} a_n(x,x).$$

The divergent terms in $W^{(1)}$ are proportional to the first coefficients $a_n(x,x)$. For even-dimensional spaces, the most important coefficient is $a_{N/2}(x,x)$, since this single coefficient for a given theory determines various anomalies [10].

In this letter, we explicitly calculate the coefficients $a_n(x,x)$ for two- and three-dimensional spaces obtained from the metric deformation of two- and three-dimensional spheres, respectively. We obtain the coefficients a_n as functions of the deformation parameters and show that under some deformation the conformal anomaly is cancelled for free scalar fields defined on $\tilde{S}^2 \times R^4$ and $\tilde{S}^2 \times R^6$.

Let us begin with the scalar Laplacian eigenvalues on deformed spheres. The metric on the deformed sphere $\tilde{S}^{d+1}$ can be expressed in the form

$$\mathrm{d}s^2 = \mathrm{d}x_0^2 + \sin^2 x_0 \, \mathrm{d}\Omega^2$$

where $\mathrm{d}\Omega^2$ is the metric on the (deformed) $\tilde{S}^d$. Any scalar function can be represented as a sum of eigenfunctions $Y_{(l)}(x_i)$ of the Laplace operator on $\tilde{S}^d$

$$\phi(x_0, x_i) = \sum_{(l)} f_{(l)}(x_0) Y_{(l)}(x_i). \tag{1}$$

Substituting decomposition (1) in the eigenvalue equation

$$\Delta\phi = \lambda\phi$$

we obtain the following ordinary differential equation

$$\left[\partial_0^2 + d\cot x_0 \partial_0 - \frac{a_{(l)}}{\sin^2 x_0}\right] f_{(l)} = \lambda f_{(l)}. \tag{2}$$

$-a_{(l)}$ is the eigenvalue of the Laplace operator on $\tilde{S}^d$ corresponding to $Y_{(l)}$. We shall drop the subscripts (l) for a while. Let us make the substitution

$$f = h\sin^b(x_0) \qquad b = \tfrac{1}{2}\left(1 - d + \sqrt{(1-d)^2 + 4a}\right)$$

and change the independent variable

$$z = \tfrac{1}{2}(\cos x_0 + 1).$$

Equation (2) then takes the form

$$z(z-1)h'' + (1+c)(z - \tfrac{1}{2})h' + eh = 0$$

$$e = b(b+d) + \lambda \qquad c = 2b + d. \tag{3}$$

Primes denote differentiation with respect to z. According to the general prescription [11], let us express h as the power series

$$h(z) = \sum_{k=0} \alpha_k z^k. \tag{4}$$

Substitution of (4) into (3) gives a recurrent condition on the coefficients α_k:

$$\alpha_{k+1} = \alpha_k \frac{k(k-1)+(1+c)k+e}{(k+1)(k+(c+1)/2)}. \tag{5}$$

The denominator of (5) is positive for all k. The eigenfunctions h_k can be found by imposing the condition that the numerator of (5) be equal to zero. We obtain the eigenvalues

$$\lambda_{(l)k} = -k^2 - (1+q)k - \tfrac{1}{2}(1-d+q+2a_{(l)})$$
$$q = \sqrt{(d-1)^2 + 4a_{(l)}} \tag{6}$$

where we have restored the dependence on the index (l). The eigenvalues $a_{(l)}$ can be defined using the same formula (6) with $d \to d-1$. Repeating these steps, we can obtain the spectrum of the scalar Laplace operator on $\tilde{S}^{d+1}$ in terms of $d+1$ non-negative integers and $d+1$ scale parameters.

For $d = 3$, equation (6) is obtained in [12] by the same methods.

In the case of the unit round d-sphere $\tilde{S}^d$ with $a_{(l)} = l(l+d-1)$, we obtain from (6)

$$\lambda_{(l)k} = -(k+l)(k+l+d) = -n(n+d) \qquad n = k+l.$$

Thus, equation (6) reproduces the correct eigenvalues of the scalar Laplace operator on the unit round S^{d+1}. One can also verify that the degeneracies have the correct values.

With the deformation of a two-dimensional sphere, we consider rescaling $l^2 \to \rho l^2$, $(\rho > 0)$, where l^2 are the eigenvalues of a Laplace operator on the unit sphere S^1. The eigenvalues (6) for $\tilde{S}^2$ can be written as

$$\lambda_{l,k} = -(k+\rho l+\tfrac{1}{2})^2 + \tfrac{1}{4}. \tag{7}$$

The heat kernel for the eigenvalues (7) is defined as

$$K(t) = K_1(t) + K_2(t) = \mathrm{e}^{t/4}\left(2\sum_{l=1}^{\infty}\sum_{k=\rho l+1/2}^{\infty} \mathrm{e}^{-k^2 t} + \sum_{k=1/2}^{\infty} \mathrm{e}^{-k^2 t}\right). \tag{8}$$

To derive the asymptotic expansion for the first term in (8), we rewrite the sum over k by using the Mellin transform

$$f(s,t) = \int_0^\infty \mathrm{d}x\, x^{s-1}\mathrm{e}^{-x^2 t} = \tfrac{1}{2}\Gamma(s/2)t^{-s/2}.$$

Performing the inverse transform

$$\frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \mathrm{d}s'\, k^{-s'} f(s',t)$$

and summing over k, we obtain

$$K_1(t) = \mathrm{e}^{t/4}\frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}s' \sum_{l=1}^{\infty}\Gamma(s'/2)t^{-s'/2}\zeta(s', \rho l+\tfrac{1}{2}) + R(t). \tag{9}$$

Here the contour C covers the poles of $\Gamma(s'/2)$ at points $s' = -2m$ as well as the poles of $g(s') = \sum_{l=1}^{\infty} \zeta(s', \rho l + \frac{1}{2})$ and

$$R(t) = \mathrm{e}^{t/4} \frac{1}{2\pi \mathrm{i}} \int_D \mathrm{d}s' \Gamma(s'/2) t^{-s'/2} g(s')$$

where the contour D consists of the semicircumference at infinity on the left. Formula (9) is understood to be exact, but it is difficult to compute $R(t)$ explicitly. However, one can show that $R(t)$ vanishes exponentially as $t \to 0$. Thus, for small t, one can discard $R(t)$ relative to the power series, leaving the asymptotic expansion for $K(t)$. (The calculations of $R(t)$ for some series can be found in [13].) Using the Hermite formula [11]

$$\zeta(z, q) = \frac{q^{-z}}{2} + \frac{q^{1-z}}{z-1} + 2\int_0^{\infty} \mathrm{d}x \, \sin(z \tan^{-1}(x/q)) \frac{(q^2 + x^2)^{-z/2}}{\mathrm{e}^{2\pi x} - 1}$$

for $\zeta(s', \rho l + \frac{1}{2})$ in (9), after summing over l and integrating over s', we obtain the following heat kernel expansion:

$$K_1(t) = \mathrm{e}^{t/4} \Bigg(\frac{1}{\rho t} - \frac{\pi^{1/2}}{2t^{1/2}} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \bigg(-\frac{2}{2m+1} \rho^{2m+1} \zeta(-2m-1, 1 + 1/(2\rho)) + \rho^{2m} \zeta(-2m, 1 + 1/(2\rho)) - \frac{2}{2m+1} \rho^{-2m-1} \zeta(-2m-1) + F(-2m, \rho) \bigg) \tag{10}$$

where

$$F(z, \rho) = 2 \sum_{p=0}^{\infty} (-1)^{p+1} c_p(z) \sum_{n=0}^{\infty} \frac{\Gamma(n + z/2)}{\Gamma(z/2) n!} \rho^{-2p-2n-z-1} \zeta(2p + 2n + z + 1, 1 + 1/(2\rho)) \times \zeta(-2p - 2n - 1) \qquad (2p + 2n + z \neq 0)$$

and the coefficients c_p are determined from

$$\sin(z \tan^{-1}(x)) = \sum_{p=0}^{\infty} c_p(z) x^{2p+1}.$$

The asymptotic expansion for K_2 in (8) can be derived by using the same method. After a little calculation (discarding the exponentially small contribution), we find

$$K_2(t) = \mathrm{e}^{t/4} \frac{\pi^{1/2}}{2t^{1/2}} \qquad (t \to 0). \tag{11}$$

Substituting (10) and (11) into (8) and performing a numerical computation, we obtain the following values for some $a_n(\rho)$ ($a_0 = 1$):

n	$\rho = 0.2$	$\rho = 0.6$	$\rho = 1$	$\rho = 1.8$
1	0.1733	0.2267	0.3333	0.7067
2	0.0077	0.0263	0.0667	0.2439
3	−0.0016	0.0024	0.0127	0.0902
4	−0.0008	0.0003	0.0032	0.0590

(12)

For $\rho = 1$, we have from (12), in a numerical form, the famous asymptotic expansion for unit round S^2

$$K(t) = \frac{1}{t} + 0.3333 + 0.0667t + 0.0127t^2 + 0.0032t^3 + \cdots .$$

The next space we would like to consider is a three-sphere with another homogeneous deformation which can be represented as $SU(2) \times U(1)/U(1)$ (the Taub space). The eigenvalues of the Laplace operator can be written as [14]

$$\lambda_{n,j} = n^2 - 1 + \omega(2j - n + 1)^2 \tag{13}$$

where ω is the deformation parameter. The range of ω is $-1 < \omega < \infty$ and $\omega = 0$ corresponds to round S^3. Then the heat kernel takes the form

$$K(t) = \sum_{n=1}^{\infty} n \sum_{j=0}^{n-1} \exp(-\lambda_{n,j})t. \tag{14}$$

First we rewrite the sum over j using the identity

$$\sum_{j=0}^{n-1} \exp(-\omega(2j - n + 1)^2)t = \left(\sum_{j=-(n-1)/2}^{\infty} - \sum_{(n+1)/2}^{\infty} \right) e^{-4\omega j^2 t}.$$

Now it has a form similar to (8) and can be evaluated by means of the Mellin transform. A straightforward calculation gives

$$K(t) = e^t \sum_{k=0}^{\infty} \frac{\omega^k (-1)^k (2k)!}{k!} \sum_{r=0}^{2k} \frac{B_r 2^r}{r!} \sum_{p=0}^{k-[(r+1)/2]} \sum_{n=1}^{\infty} \frac{e^{-n^2 t} n^{2p+2} t^k}{(2k - 2p - r)!(2p + 1)!}. \tag{15}$$

Here we have used the representation

$$\zeta(-m, q) = -\sum_{r=0}^{m+1} \frac{m! B_r q^{m+1-r}}{r!(m - r + 1)!}$$

where B_r are Bernoulli numbers. After similar manipulations with the sum over n in (15), we obtain

$$\begin{aligned} K(t) &= e^t \sum_{k=0}^{\infty} \frac{\omega^k (-1)^k (2k)!}{4k!} \sum_{r=0}^{2k} \frac{B_r 2^r}{r!} \sum_{p=0}^{k-[(r+1)/2]} \frac{\Gamma(3/2 + p) t^{k-p-3/2}}{(2k - 2p - r)!(2p + 1)!} \\ &= \frac{\pi^{1/2}}{4(1+\omega)^{1/2}} \left(1 + \frac{3 + 4\omega}{3(1+\omega)} + \frac{32\omega^2 + 40\omega + 15}{30(1+\omega)^2} \right. \\ &\quad \left. + \frac{369\omega^3 + 28\omega^2 + 140\omega + 35}{210(1+\omega)^3} + \cdots \right). \end{aligned} \tag{16}$$

With $\omega = 0$, the expansion for round S^3 is reproduced.

As is known, the divergences in the one-loop effective action for even-dimensional spaces lead to scale symmetry breaking and give rise to a non-vanishing conformal anomaly.

The conformal anomaly has a geometric structure and is expressed by means of $a_{N/2}$. In our case, a_n depend on the deformation parameters and can be equal to zero with the appropriate parametric values.

Let us consider the one-loop effective action for scalar fields on $R^m \times \tilde{S}^2$ where R^m is Euclidean m-dimensional space. The conformal anomaly arises when we take the expectation value of the momentum–energy tensor T^μ_μ with the metric as a background classical field

$$\langle T^\mu_\mu \rangle = \frac{g_{\mu\nu}}{Z[g]} \frac{\delta Z[g]}{\delta g_{\mu\nu}}$$

where $Z[g]$ is the generating functional of the theory. Zeta-function regularization gives

$$\langle T^\mu_\mu \rangle = \frac{1}{(4\pi)^{(m+2)/2}} a_{(m+2)/2}. \tag{17}$$

From (10), (11) and (8), one can compute that the anomaly (17) for scalar fields on $R^4 \times \tilde{S}^2$ and $R^6 \times \tilde{S}^2$ is removed with the values $\rho = 0.41$ and $\rho = 0.51$, respectively. The Casimir energy is finite for these spaces and can be computed explicitly. (This problem is now under consideration.) For scalar fields on the four-dimensional space $R^1 \times SU(2) \times U(1)/U(1)$, the anomaly

$$\langle T^\mu_\mu \rangle = \frac{1}{(4\pi)^2} \frac{32\omega^2 + 40\omega + 15}{30(1+\omega)^2}$$

cannot be removed with any value of ω.

It should be noted that a different type of deformed sphere has been considered previously in multidimensional models [15]. However, only small one-parameter deformations have been used for the calculation of the one-loop potential. In our case, the deformation removing the conformal anomaly cannot be considered small.

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Note added. The manifolds with singular points were also studied in the context of orbifold factors of spheres and flat conical spaces. The corresponding references can be found in [16, 17]. One of us (DV) is grateful to Guido Cognola for pointing out [17].

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